Residuo

Sea la función $f\colon \mathbb{Z} \to \{0,1\}$ definida así:

$$f(n) = \begin{cases} 0, & n \equiv 0 \ (mod\ 2) \\ 1, & n \equiv 1 \ (mod\ 2) \end{cases}$$

$$f(n) = \frac{1-(-1)^n}{2}$$

Cociente entero

Sea la función $g\colon \mathbb{Z} \to \mathbb{Z}$ definida así:

$$g(n) = \begin{cases} \dfrac{n}{2}, & n \equiv 0 \ (mod\ 2) \\ \dfrac{n-1}{2}, & n \equiv 1 \ (mod\ 2) \end{cases}$$

$$g(n) = \frac{n-f(n)}{2} = \frac{2n-1+(-1)^n}{4} = \left\lfloor \frac{n}{2} \right\rfloor$$

Ejemplo

Para $x \in \mathbb{R}$ y $n \in \mathbb{N} \cup \{0\}$:

$$\frac{d^n}{dx^n}\cos x = (-1)^{g(n+1)}\big(f(n+1)\cos x + \big(1-f(n+1)\big)\sin x\big)$$

$$\frac{d^n}{dx^n}\sin x = (-1)^{g(n)}\big(f(n)\cos x + \big(1-f(n)\big)\sin x\big)$$

Fórmulas de ángulo múltiple para el coseno

Para $n \in \mathbb{N} \cup \{0\}$:

$$\cos nx = \sum_{k=0}^{g(n)} (-1)^k \binom{n}{2k} (\cos x)^{n-2k} (\sin x)^{2k}$$

$$= \sum_{k=0}^{g(n)} \binom{n}{2k} \left(\sum_{p=0}^{k} (-1)^p \binom{k}{p} (\cos x)^{n-2p} \right) = \sum_{k=0}^{g(n)} (-1)^k (\cos x)^{n-2k} \left(\sum_{p=k}^{g(n)} \binom{n}{2p} \binom{p}{k} \right)$$

Luego,

$$\sum_{n=0}^{m} \cos nx = \sum_{n=0}^{m} (\cos x)^n \left(\sum_{k=0}^{g(m-f(n))-g(n)} (-1)^k \left(\sum_{p=k}^{g(2k+n)} \binom{2k+n}{2p} \binom{p}{k} \right) \right) = \frac{\sin \frac{(m+1)x}{2}}{\sin \frac{x}{2}} \cos \frac{mx}{2}$$

Nota:

$$\sum_{n=0}^{m} \left(\sum_{k=0}^{g(n)} a_k b_{n-2k} c_{n,k} \right) = \left[\sum_{n=0}^{g(m)} b_{2n} \left(\sum_{k=0}^{g(m)-n} a_k c_{2k+2n,k} \right) \right] + \left[\sum_{n=0}^{g(m-1)} b_{2n+1} \left(\sum_{k=0}^{g(m-1)-n} a_k c_{2k+(2n+1),k} \right) \right] = \sum_{n=0}^{m} b_n \left(\sum_{k=0}^{g(m-f(n))-g(n)} a_k c_{2k+n,k} \right)$$

$$\sum_{m=0}^{q}(\cos mx)^n=\frac{1}{2^{n-1}}\left(\frac{\sin\frac{(q+1)nx}{2}}{\sin\frac{nx}{2}}\cos\frac{qnx}{2}-\sum_{k=1}^{g(n)}(-1)^k\left(\sum_{p=k}^{g(n)}\binom{n}{2p}\binom{p}{k}\right)\left(\sum_{m=0}^{q}(\cos mx)^{n-2k}\right)\right)$$

Fórmulas de ángulo múltiple para el seno

Para $n\in\mathbb{N}$:

$$\sin nx=\sum_{k=0}^{g(n-1)}(-1)^k\binom{n}{2k+1}(\cos x)^{n-(2k+1)}(\sin x)^{2k+1}$$

$$=(\cos x)^{f(n-1)}\sum_{k=0}^{g(n-1)}\binom{n}{2k+1}\left(\sum_{p=k}^{g(n-1)}(-1)^p\binom{g(n-1)-k}{g(n-1)-p}(\sin x)^{2p+1}\right)$$

$$=(\cos x)^{f(n-1)}\sum_{k=0}^{g(n-1)}(-1)^k(\sin x)^{2k+1}\left(\sum_{p=0}^{k}\binom{n}{2p+1}\binom{g(n-1)-p}{g(n-1)-k}\right)$$

Luego,

$$\sum_{n=1}^{m}\sin nx=\sum_{n=0}^{g(m-1)}(-1)^n(\sin x)^{2n+1}\left(\sum_{k=2n+1}^{m}(\cos x)^{f(k-1)}\left(\sum_{p=0}^{n}\binom{k}{2p+1}\binom{g(k-1)-p}{g(k-1)-n}\right)\right)=\frac{\sin\frac{(m+1)x}{2}}{\sin\frac{x}{2}}\sin\frac{mx}{2}$$

Si además $n\in\{0\}$:

$$\sin(2n+1)x=\sum_{k=0}^{n}(-1)^k(\sin x)^{2k+1}\left(\sum_{p=0}^{k}\binom{2n+1}{2p+1}\binom{n-p}{n-k}\right)\qquad(*)$$

Por consiguiente,

$$\sum_{n=0}^{g(m)}\sin(2n+1)x=\sum_{n=0}^{g(m)}(-1)^n(\sin x)^{2n+1}\left(\sum_{k=n}^{g(m)}\left(\sum_{p=0}^{n}\binom{2k+1}{2p+1}\binom{k-p}{k-n}\right)\right)=\frac{(\sin g(m)x)^2}{\sin x}$$

$$\sum_{m=0}^{q}(\sin mx)^{2n+1}=\frac{(-1)^n}{2^{2n}}\left(\frac{\sin\frac{(q+1)(2n+1)x}{2}}{\sin\frac{(2n+1)x}{2}}\sin\frac{q(2n+1)x}{2}-\sum_{k=0}^{n-1}(-1)^k\left(\sum_{p=0}^{k}\binom{2n+1}{2p+1}\binom{n-p}{n-k}\right)\left(\sum_{m=0}^{q}(\sin mx)^{2k+1}\right)\right)$$

De $(*)$, si $x=\frac{\pi}{2n+1}$ y $n>0$, entonces:

$$\sum_{k=0}^{n}(-1)^k\left(\sin\frac{\pi}{2n+1}\right)^{2k}\left(\sum_{p=0}^{k}\binom{2n+1}{2p+1}\binom{n-p}{n-k}\right)=0$$

Donde $\sin\frac{\pi}{2n+1}\neq 0$

Por ejemplo, para $n = 2$, se obtiene la expresión $16\left(\sin\frac{\pi}{5}\right)^4 - 20\left(\sin\frac{\pi}{5}\right)^2 + 5 = 0$, donde $\sin\frac{\pi}{5} = \frac{1}{2}\sqrt{\frac{5-\sqrt{5}}{2}}$ (teniendo en cuenta que, en el primer cuadrante, $\sin\frac{\pi}{6} < \sin\frac{\pi}{5} < \sin\frac{\pi}{4}$).

Serie

Para $p \in \mathbb{N}$ y $|x| < 1$:

$$\sum_{k=1}^{n} k^p\, x^{k-1} = \frac{1}{1-x}\left(\sum_{j=0}^{p-1}\binom{p}{j}\left(\sum_{k=0}^{n-1} k^j\, x^k\right) - n^p x^n\right)$$

Donde $\sum_{k=0}^{n-1} x^k = \frac{1-x^n}{1-x}$